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Narrow Compactness of $\mathcal{C}_c$

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ABSTRACT

Let Ω be a Polish space. A Choquet capacity defined on the sample space Ω is generalization of a probability measure which is subadditive in its argument. This paper presents some important convergence properties concerning Choquet capacities. In particular, a Portmanteau-type theorem for Choquet capacities is presented. Also, important properties of Choquet integrals are given to aid in characterizing some convergence results. Ultimately, this paper shows the compactness of the space of Choquet capacities, $\mathcal{C}_c$, topologize by the narrow topology in the sense of O'Brien(1996), when the sample space Ω is compact.

Keywords and phrases: Choquet capacities, two-alternating, upper probabilities, Choquet integral, comonotonic functions, narrow topology, compact metric spaces.

I. INTRODUCTION

Let Ω be a Polish space (*metrizable metric space which is separable and complete*) and let F be a collection of subsets of Ω containing ∅ and closed under finite union and intersection. Let 2^Ω bet the power set of Ω. A Choquet capacity $v:2^\Omega \to [-\infty,\infty]$ is a set function satisfying the following properties: (a) $v(A)\le v(B)$ whenever $A\subset B$; (b) for every increasing sequence $\{A_n\}\subset 2^\Omega$,

$$v\left(\bigcup_{n=1}^{\infty} A_n\right) = \sup_{n\in \mathrm{N}} v(A_n) \tag{1}$$

(c) for every decreasing sequence $\{A_n\}$ of element of F ,

$$v\left(\bigcap_{n=1}^{\infty} A_n\right) = \inf_{n\in N} v(A_n) \tag{2}$$

Notice that probability measures are special cases of Choquet capacities. A capacity v is said to be *two-alternating* if $v(A\cap B)+v(A\cup B)\le v(A)+v(B)$ whenever $A\subset B$. A capacity is *coherent* if, and only if, there is a family P of probability measures defined on Ω such that $v(A)=\sup_{P\in \mathrm{P}} P(A)$ for any A in the Borel σ − field B of Ω. Coherent capacities are also known as *upper probabilities.*

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Let $C_b(\Omega)$ be the vector space of all bounded and continuous real-valued functions on Ω and let c be a Choquet capacity. Define the *Choquet integral* of $X \in C_b(\Omega)$ relative to c as

$$\int X\,dc = \int_0^\infty c(\tilde{X} > t)\,dt = \int_0^\infty c(\tilde{X} \geq t)\,dt$$

where $\tilde{X} = X + b$ and b is a constant chosen to make $X + b \geq 0$. We denote this by $\tilde{c}(X)$. Choquet integrals satisfy the following properties:

(P1.) $\tilde{c}(X) \geq 0$ for all $X \in C_b(\Omega)$.

(P2.) $\tilde{c}(X) \leq \tilde{c}(Y)$ whenever $X \leq Y$.

(P3.) For any constant $a > 0$, $\tilde{c}(aX) = a\tilde{c}(X)$.

(P4.) c is two-alternating if, and only if, $\tilde{c}(X+Y) \leq \tilde{c}(X) + \tilde{c}(Y)$.

This paper focuses mainly on the compactness of the space of two-alternating upper probabilities, denoted by $\mathcal{C}_c$, with the respect to the narrow topology. In the proof of the main theorem, it will be seen that the sample space Ω must be compact. This assumption seems to be too restrictive but this is necessary so that all assumptions, concerning capacities in robust statistics, are fulfilled for neighborhoods of probability measures in terms of $\varepsilon -$contamination and total variation (Rieder, et al., 1977).

II. THE NARROW TOPOLOGY

Let $\mathcal{C}_c$ be the space of two-alternating coherent Choquet capacities defined on the space $(\Omega, \mathcal{B})$, where $\mathcal{B}$ is the Borel $\sigma -$field on Ω. We topologize this space in the sense of O'Brien (1996).

Definition 1. The narrow topology on $\mathcal{C}_c$ is the topology generated by the subbase consisting of all subsets of the form $\{v \in \mathcal{C}_c : v(G) > x\}$ or $\{v \in \mathcal{C}_c : v(F) < x\}$, where G is open, F is closed, and $x \in (0,1)$.

Based from the above definition of narrow topology, we have the following definition of *narrow convergence.*

Definition 2. Let $\{v_n\}$ be a sequence of capacities in $\mathcal{C}_c$. Then we say that v_n *converges narrowly* to v, denoted by $v_n \xrightarrow{narrow} v$, if and only if

$$\begin{aligned} \liminf v_n(G) &\geq v(G) \\ \limsup v_n(F) &\leq v(F) \end{aligned} \tag{3}$$

for all G open and F closed sets.

Note that this definition is analogous to the definition of weak convergence of probability measures. See Parthasarathy (1967) and Billingsley (1999) for the detailed account on weak convergence.

Let $\{P_1, P_2, \ldots, P\}$ be (weakly) compact and convex subsets of M and, for each *n*, define $v_n(A) = \sup_{P \in P} P(A)$. Let ∂A denote the boundary of the set A and *int*(A) be the interior of A. The next theorem characterizes narrow convergence of two-alternating coherent Choquet capacities. For the proof, see Bataller (2003).

Theorem 1. The following are equivalent:

1. $v_n \xrightarrow{narrow} v$
2. $v_n(A) \to v(A)$ for all measurable set A such that $v(\partial A) = 0$.
3. $\tilde{v}_n(X) \to \tilde{v}(X)$ for all bounded, positive and continuous functions X in Ω.

The above theorem is analogous to Portmanteau theorem for the theory of weak convergence of probability measures. See Billingsley (1999) and Parthasarathy (1967).

We will define a special class of functions called *comonotonic functions.* These functions play a very significant role in the proof of compactness of C_c. Let $C \subset C_b(\Omega)$. We say that C is *comonotonic* if, and only if, for each pair of $X, Y \in C$, there are no $\omega_1, \omega_2 \in \Omega$. The following result is a special case of Proposition 3.14 of Boccuto and Sambucini (1996).

Theorem 2. Let v be a Choquet capacity satisfying $v(\varnothing) = 0$. If $C \subset C_b(\Omega)$ is comonotonic, then $\tilde{v}(X+Y) = \tilde{v}(X) + \tilde{v}(Y)$ for all $X, Y \in C$.

From the above theorem, Choquet integrals are linear whenever the functions belong to a comonotonic class.

III. FOUR LEMMAS

We start the section by stating a lemma due to Huber and Strassen (1973).

Lemma 1. Let v be a Choquet capacity and let $C_b^+(\Omega)$ be the space of bounded and continuous non-negative real functions. The relation $\tilde{P}(X) = \int X dP$ defined a one-to-one correspondence between the probability measures $P \le v$ and the positive linear functionals $\tilde{P}$ on $C_b(\Omega)$ satisfying $\tilde{P}(1) = 1$ and $\tilde{P}(X) \le \tilde{v}(X)$ for all $X \in C_b^+(\Omega)$.

Using Lemma 1, we have the following result which is a consequence of Hahn-Banach theorem.

Lemma 2. Let v be a two-alternating upper probability generated by some weakly compact and convex set of probability measures P. Then for every class C of non-negative, bounded, continuous, and comonotonic functions, there exists a probability measures $P \in \mathrm{P}$ such that $P \leq v$ and $\tilde{v}(X) = \int XdP$ for all X in C.

Proof: By Theorem 2, the functional Λ defined by $\Lambda(X) = \tilde{v}(X)$, for some two-alternating upper probability v, is linear and bounded for all $X \in C$. Define the bounded linear functional $\tilde{P}(X) = \int XdP$, for some probability measure P on Ω and $X \in C_b(\Omega)$. Now by Lemma 1, there always a probability measure $P \in \mathrm{P}$, where P is generated by a two-alternating upper probability v, such that $P \leq v$ and $\tilde{P}(X) \leq \tilde{v}(X)$ for all $X \in C_b(\Omega)$ such that $X \geq 0$. Thus by the Hahn-Banach theorem, the functional Λ can be extended to $\tilde{P}$, and so $\tilde{v}(X) = \int XdP$ for all $X \in C$. ■

The next lemma was due to Parthasarathy (1967).

Lemma 3. If Ω is a totally bounded metric space, then $U(\Omega)$, the space of bounded and uniformly continuous real functions on Ω, is a separable Banach space under the norm

$$\|X\|_{\infty} = \sup_{\omega \in \Omega} |X(\omega)| \tag{4}$$

Lemma 4. Let Ω be a compact metric space and $U(\Omega)$ be given. Suppose there are some $v, v' \in \mathcal{C}_c$ such that for all $X \in U(\Omega)$. Then, $v = v'$.

Proof: For any closed set F, there exists a function $X : \Omega \to \mathbb{R}$ such that

$$1_F(\omega) \leq X(\omega) = \left(1 - \rho(\omega, F)/\varepsilon\right)^{+} \leq 1_{F^{\varepsilon}}(\omega)$$

where ρ is the metric on Ω and ε is an arbitrary positive constant (Billingsley (1999). Note that X is bounded and uniformly continuous. Thus we have

$$v(F) = \tilde{v}(1_F) \leq \tilde{v}(X) = \tilde{v}'(X) \leq \tilde{v}'(1_{F^{\varepsilon}}) = v'(F^{\varepsilon})$$

Letting $\varepsilon \downarrow 0$, we have $v(F) \leq v'(F)$. Interchanging v and v' in the above argument we get $v'(F) \leq v(F)$. Thus $v(F) = v'(F)$ for all closed set F. Since Ω is compact, then each F is also compact. Thus the regularity of v implies $v = v'$. See Bataller (2003). ◽

IV. THE MAIN RESULT

We are now ready to prove the main result of this paper.

Theorem 3. Let Ω be a compact metric space. Then the space C_c can be metrized as a compact metric space.

Proof: Let Ω be a compact metric space. Then Ω is totally bounded and so by Lemma 3, $U(\Omega)$ is a separable Banach space under the sup norm given by (3). Let U be a subset of $U(\Omega)$ that is comonotonic and consider the set $H \subset U$ defined by

$$H = \{X \in U : X \geq 0, \|X\|_\infty < 1\}$$

The separability of $U(\Omega)$ implies the separability of H. Let $\{X_1, X_2, \ldots\}$ be a countable dense subset of H. Let I^∞ be the countable product of the interval [0,1]. Note that I^∞ is a compact metric space (Aliprantis and Border (1994). Define the map $T : C_c \to I^\infty$ as follows: for each $v \in C_c$, $T(v) = \{\tilde{v}(X_1), \tilde{v}(X_2), \ldots, \tilde{v}(X_n), \ldots\}$. First we will show that T is a homeomorphism. To show T is one-to-one, suppose $T(v) = T(v')$. Then $\tilde{v}(X_r) = \tilde{v}'(X_r)$ for each r. Since $\{X_1, X_2, \ldots\}$ is dense in H, then $\tilde{v}(X) = \tilde{v}'(X)$ for all X in H. Hence by Lemma 4, $v = v'$. To show that T is continuous, suppose $v_n \xrightarrow{narrowly} v$. Then $\tilde{v}_n(X_r) \to v(X_r)$ for each r. This implies that $T(v_n) \to T(v)$. Lastly to show that T^{-1} is continuous, let $\{v_n\}$ be a sequence in C_c and let $T(v_n) \to T(v)$. Let $X \in H$ such that $\|X - X_{r_k}\| \to 0$ as k becomes large for some subsequence $\{X_{r_k}\} \subset \{X_r\}$. The linearity of $\tilde{v}$ follows from Theorem 2, and so,

$$\begin{aligned}
|\tilde{v}_r(X) - \tilde{v}(X)| &= |\tilde{v}_n(X - X_{r_k} + X_{r_k}) - \tilde{v}(X - X_{r_k} + X_{r_k})| \\
&\leq |\tilde{v}(X - X_{r_k})| + |\tilde{v}(X - X_{r_k})| + |\tilde{v}_n(X_{r_k}) - \tilde{v}(X_{r_k})| \\
&\leq 2\|X - X_{r_k}\|_\infty + |\tilde{v}_n(X_{r_k}) - \tilde{v}(X_{r_k})|
\end{aligned}$$

Thus as both n and k become large, $|\tilde{v}_n(X) - \tilde{v}(X)| \to 0$. Hence by Theorem 1, $v_n \xrightarrow{narrowly} v$. This shows that T is a homeomorphism. To end this proof, we need to show that $T(C_c)$ is closed. Now suppose $\{v_n\} \subset C_c$ be a sequence such that $T(v_n)$ converges to $(d_1, d_2, \ldots)$. Let $X \in H$. Then there exists a sequence $\{X_{r_k}\}$ such that $\|X - X_{r_k}\| \to 0$ as $k \to \infty$. Then we have

$$|\tilde{v}_n(X) - \tilde{v}_m(X)| \leq 2\|X - X_{r_k}\|_\infty + |\tilde{v}_n(X_{r_k}) - \tilde{v}_m(X_{r_k})|$$

Thus as *m,n,k* goes large, we have $\left|\tilde{v}_n(X)-\tilde{v}_m(X)\right|\to 0$. Hence for each *X* in *H*, $\{\tilde{v}_n(X)\}$ is a Cauchy sequence, and so $\lim\tilde{v}_n(X)$ exists. Let $\Lambda(X)=\lim\tilde{v}_n(X)$. Note that for any *g* in *U*, we can find $c\neq 0$ such that *cg* in *H*. Define the functional $\Lambda(g)=c\Lambda(g/c)$. From the construction, Λ is positive, bounded, and linear defined on *U* such that $\Lambda(1)=1$. Hence by Lemma 1, there exists a $P\in\mathrm{P}$, where P is some family of probability measures defined on Ω, such that $P\leq v$, where $v=\sup_{P\in\mathrm{P}}P$, and $\Lambda(X)=\int XdP$. By Lemma 2, this implies that $\Lambda(X)=\tilde{v}(X)$. In particular, $\tilde{v}_n(X_r)=d_r$. Thus, $T(v)=(d_1,d_2,\ldots)$. In other words, $T(C_c)$ is closed, and hence the compactness of I^{∞} implies that C_c is a compact metric space. This ends the proof. ■

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